LAWVERE'S FIXED POINT THEOREM

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These notes explain Lawvere's fixed point theorem from scratch and go over some of its applications.

1. Cartesian Closed Categories

We'll start at the beginning with a definition of a category.

Definition 1.1. *A* ***category*** $\mathcal{C}$ *consists of a class* $\mathsf{ob}\mathcal{C}$ *of* ***objects*** *and for any two objects* A, B *in* $\mathsf{ob}\mathcal{C}$ *a set* $\mathsf{Hom}_{\mathcal{C}}(A, B)$ *of* ***morphisms*** $f : A \to B$ *such that*

- *for each object* $A \in \mathsf{ob}\mathcal{C}$ *there exist a morphism* $1_A : A \to A$ *called the* ***identity morphism****,*
- *for any morphisms* $f : A \to B$ *and* $g : B \to C$*, there exists a morphism* $g \circ f : A \to C$ *called the* ***composition*** *of* f *and* g*,*
- *for each morphism* $f : A \to B$*,*

$$1_B \circ f = f = f \circ 1_A$$

  and
- *for any morphisms* $f : A \to B$*,* $g : B \to C$ *and* $h : C \to D$*,*

$$h \circ (g \circ f) = (h \circ g) \circ f$$

  .

A morphism $f : A \to B$ is also called a ***map*** or ***arrow*** from A to B. We say that A is the ***domain*** of the morphism f and that B is the ***codomain*** of f.

There are many examples of categories in mathematics. A salient example is the category **Set**. The objects of this category are sets. A morphism $f : A \to B$ is simply a function whose domain is A and whose codomain is B. The function $1_A : A \to A$ is that function that sends each element of A to itself (i.e., the identity function).[1]

Another class of examples of categories consists of sets with some structure built on them. The morphisms are then structure preserving functions between the objects in question. So there is a category **Grp** whose objects are groups and whose morphisms are group homomorphism, a category **Ring** whose objects are rings and whose morphisms are ring homomorphisms, and a category **Top** whose objects are topological spaces and whose morphisms are continuous maps. More generally, for any first-order signature $\mathcal{L}$ there is a category who objects are $\mathcal{L}$-structures and whose morphisms are homomorphisms between $\mathcal{L}$-structures.[2]

These are all examples of *concrete categories*. The morphisms of the category are functions of certain kinds and the objects are sets, possibly with some distinguished structure. But there are important examples of non-concrete categories that illustrate some of the below definitions nicely. We will focus on one example in particular: ordered sets.

Let X be a set. A ***preorder*** on X is a binary relation $R \subseteq X \times X$ that is reflexive and transitive. A set equipped with a preorder is called a ***preordered set***. Given any preordered set X (with preorder R) we can define an associated category $C(X)$ as follows. We let $\mathsf{ob}C(X) = X$ and for each $x, y \in X$, we let we let there be a morphism $f : x \to y$ if and only if $(x, y) \in R$ (we could simply let the pair (x, y) be this morphism and let $(y, z) \circ (x, y) = (x, z)$). Conversely, let $\mathcal{C}$ be any category such that $\mathsf{ob}\mathcal{C}$ is a set and between any two objects there is exactly one arrow. Then we can define a corresponding preorder R on $\mathsf{ob}\mathcal{C}$ by letting $(x, y) \in R$ if and only if there is a morphism $f : x \to y$. Thus we can regard preordered sets as categories with a set of objects that are such that, between any two objects there is exactly one arrow.

[1]Altough not strictly required, it is common to regard the sets $\mathsf{Hom}_{\mathcal{C}}(A, B)$ and $\mathsf{Hom}_{\mathcal{C}}(A, D)$ as being disjoint whenever B and D are different objects. In the case of **Set** this requires the Bourbakian notion of function according to which each function uniquely determines a codomain (one can simply take functions in **Set** $f : A \to B$ to be triples $(G(f), A, B)$ where $G(f)$ is the graph of f, A is the domain of f, B is a set that includes the range of f. With this notion of function we can regard functions as being surjections or being bijections *simpliciter* rather than being surjections between some sets but not others. This allows for more symmetry in the notions of injection and surjection.

[2]We could also look at categories with the same objects but whose morphisms are elementary embeddings.

Just as we can regard preordered sets as categories, so too we can regard partially ordered sets as categories. First a definition. Let $\mathcal{C}$ be any category. We say that a morphism $f : A \to B$ is an ***isomorphism*** if there exists a morphism $g : B \to A$ such that $g \circ f = 1_A$ and $f \circ g = 1_B$. This categorical notion of ismorphism provides a very general account that subsumes "concrete" definitions of isomorphisms. For instance given any signature $\mathcal{L}$ the isomorphisms in the category of $\mathcal{L}$-structures are precisely the bijective homomorphisms. And more generally it gives us the right notion of sameness in categories. In **Top** the isomorphisms are precisely the homeomorphisms. But there are bijective morphisms in **Top** (i.e., continuous functions that are bijections), that are not homeomorphisms.

With the notion of isomorphisms in place, we can define a ***partially ordered set*** to simply be a preordered set, viewed as a category, in which isomorphic objects are identical. Since in category theory objects are really only studied up to isomorphisms, there is a precise sense in which preorders and partial orders are categorically indistinguishable. Thus from the perspective of category theory, a set F of formulas preordered by a classical consequence relation $\vdash$ is indistinguishable from the partially ordered set of equivalence classes of F under mutual provability (with the order given by $[\varphi] \leq [\psi]$ if and only if $\vdash \varphi \to \psi$).

In 1969 Lawvere showed that Cantor's theorem followed as a simple corollary of a much more general theorem concerning the class of cartesian closed categories. The category of sets is a cartesian closed category. But there are many other examples. For instance, any Heyting algebra can be viewed as a Cartesian closed category (and thus every Boolean algebra is also a cartesian closed category). In order to understand the notion of a Cartesian closed category, there are three things we need to introduce: the notion of a terminal object, the notion of a product, and the notion of an exponential object. I will explain these in turn.

The simplest notion to explain is that of a terminal object.

Definition 1.2. *Let $\mathcal{C}$ be a category. An object A in $\mathcal{C}$ is* ***terminal*** *if for every B in $\mathcal{C}$ there is a unique morphism $f : B \to A$.*

We can illustrate this definition with the category **Set**. Let $A = \{x\}$ be a singleton. Then for any set B, there will be exactly one function $f : B \to A$: the function that sends each element $y \in B$ to x. Conversely suppose that A is a set with at least two elements. For each $x \in A$, there is a function $f_x : A \to A$ that maps everything in A to x. Since A has at least two elements x and y, there are then at least two function f_x and f_y from A to A. Hence A is not terminal. The result is that in the category **Set**, the terminal objects are precisely the singletons.

In concrete categories, terminal objects look similar to the terminal objects in **Set**. In **Grp** the terminal object consists of the trivial group, whose sole element is the unit of the group. In **Top** the terminal objects consist of one point spaces. But the notion of terminal objects also makes sense in a poset. Given any poset X an object $1 \in X$ is terminal if for any $x \in X$, $x \leq 1$. Thus the terminal object of a poset is the top element of the poset. Thus a poset has a greatest element if and only if it has a terminal object.

A category may have many terminal objects (as in **Set**). But the terminal objects are always unique up to unique isomorphism. That is, if A and B are terminal in $\mathcal{C}$ there exists a unique isomorphism $f : A \to B$. To see this note that since both A and B are terminal there are unique morphisms $f : A \to B$ and $g : B \to A$. Thus $g \circ f : A \to A$ is a morphism. But since A is terminal, there is a unique morphism from A to A. And so since $1_A : A \to A$ we have that $g \circ f = 1_A$. Analogous reasoning shows $f \circ g = 1_A$. Hence f is the unique isomorphism from A to B.

The second notion appealed to in the definition of a Cartesian closed category is the notion of product.

Definition 1.3. *Let $\mathcal{C}$ be a category and let A and B be two objects in $\mathcal{C}$. A* **product** *of A and B is an object $A \times B$ of $\mathcal{C}$ together with a pair of morphisms $\pi_A : A \times B \to A$ and $\pi_B : A \times B \to B$ such that for any object Y and any morphisms $f_1 : Y \to A$ and $f_2 : Y \to B$ there exists a unique morphism $(f_1, f_2) : Y \to A \times B$ such that the following* *diagram commutes:*

$$\begin{tikzcd} & Y \arrow[dl, bend right, "f_1"'] \arrow[d, dashed, "(f_1,f_2)"] \arrow[dr, bend left, "f_2"] & \\ A & A \times B \arrow[l, "\pi_A"] \arrow[r, "\pi_B"'] & B \end{tikzcd}$$

The concept of a product can again be illustrated with the category **Set**. Let A and B be any sets and let $A \times B = \{(a,b) \mid a \in A, b \in B\}$ be the Cartesian product of A and B. Let $\pi_A : A \times B \to B$ be the projection $(a,b) \mapsto a$ and let $\pi_B : A \times B \to B$ be the projection $(a,b) \mapsto b$. It is not hard to verify then that $A \times B$ with π_A and π_B is a product in **Set**.

The notion of product applies in may other categories. In the category **Top** the product of two topological spaces is the product space. In the category **Grp** the product of two groups is the direct product of groups. We can also illustrate the definition in the case of a poset. For any two elements $x, y \in X$ of a poset X, the product $x \times y$ of x and y is that element such that, for any any $z \in X$, $z \leq x \times y$ if and only if $z \leq x$ and $z \leq y$. That is, the product of two elements is their *infimum* or *meet*. Note that it is not always the case that in a partial order two elements have a meet. Thus products needn't always exist in a category.

We'll say that a category *has binary products* if any two objects of the category have a product. Thus a poset X has binary products if and only if X is a meet-semilattice. The following notation will be useful in what follows. Let $f : A \to B$ and $g : C \to D$ be two morphisms in a category that has binary products. Then we can define the morphism $f \times g : A \times C \to B \times D$ to be $(f \circ \pi_A, g \circ \pi_C)$. That is, $f \times g$ is the unique morphism such that the following diagram commutes:

$$\begin{tikzcd} B & B \times D \arrow[l, "\pi_B"'] \arrow[r, "\pi_D"] & D \\ A \arrow[u, "f"] & A \times C \arrow[u, "f \times g"] \arrow[l, "\pi_A"] \arrow[r, "\pi_C"'] & C \arrow[u, "g"'] \end{tikzcd}$$

The existence and uniqueness of $f \times g$ follows from the definition of products. In the category of sets, $f \times g : A \times C \to B \times D$ can be defined to be that function that maps any pair $(a,c) \in A \times C$ to $(f(a), g(c)) \in B \times D$ (since $\pi_1(a,c) = a$ and $\pi_2(a,c) = c$.)

Two objects are not guaranteed to have a unique product. But like terminal objects, products are unique up to unique isomorphism. I won't give the proof here, but the idea is substantially the same as that for terminal objects.

A second important map to appeal to with respect to products is the *diagonal map* $\Delta : A \to A \times A$; this map will play an important role in the diagonalization proof of Lawvere's. This map can be defined simply as $(1_A, 1_A)$. Thus Δ is the unique map such that the following diagram commutes:

$$\begin{tikzcd} & A \arrow[dl, "1_A"'] \arrow[d, dashed, "\Delta"] \arrow[dr, "1_A"] & \\ A & A \times B \arrow[l, "\pi_A"] \arrow[r, "\pi_A"'] & A \end{tikzcd}$$

We are now in a position to introduce the final componenent of cartesian closed categories: exponentials.

Definition 1.4. *Let $\mathcal{C}$ be a category that has binary products. Let A and B be objects of $\mathcal{C}$. Then an **exponential** of A and B is an object B^A of $\mathcal{C}$ together with a morphism $ev : B^A \times A \to B$ such that for any object C and morphism $f : C \times A \to B$ there is a unique morphism $\bar{f} : C \to B^A$ (the **transpose** of f) such that the following diagram commutes:*

$$\begin{tikzcd} B^A \times A \arrow[r, "ev"] & B \\ C \times A \arrow[u, "\bar{f} \times 1_A"] \arrow[ur, "f"'] & \end{tikzcd}$$

It will be helpful again to work through some examples. The category **Set** will again illustrate the definition simply. Where A and B are two sets, the exponential of A and B is the set $B^A = \{f : A \to B\}$ of all functions from A to B. The morphism $ev : B^A \times A \to A$ is the function that takes a function $f : A \to B$ and argument $a \in A$ and evaluates f at the point a (i.e., $ev(f, a) = f(a)$).

Let $\mathcal{C}$ be any category with products and exponentials. The operation taking a morphism $f : C \times A \to B$ to its transpose $\bar{f} : C \to B^A$ is a bijection from $\mathcal{C}(C \times A, B)$ to $\mathcal{C}(C, B^A)$. To see this we can describe an inverse to this operation, also denoted by $\bar{\cdot}$ taking morphism

$f : C \to B^A$ to morphisms $\bar{f} : C \times A \to B$. We define $\bar{f}$ to be the following composite:

$$\begin{tikzcd}
B^A \times A \arrow[dr, "ev"] & \\
C \times A \arrow[u, "{(f,1_A)}"] \arrow[r, "\bar{f}"'] & B
\end{tikzcd}$$

We've thus described a map from morphisms $f : C \times A \to B$ to morphisms $\bar{f} C \to B^A$ and a map from morphisms $g : C \to B^A$ to morphisms $\bar{g} : C \times A \to B$. We'll now show that these are mutually inverse.

Proposition 1.5. *Let $\mathcal{C}$ be a category with products and exponentials. Then for any morphism $f : C \times A \to B$,*

$$\bar{\bar{f}} = f$$

And for any morphism $g : C \to B^A$,

$$\bar{\bar{g}} = g$$

Proof. Let $f : C \times A \to B$ be a morphism. Then by the definition of exponentials

$$f = ev \circ (\bar{f}, 1_A)$$

Where $\bar{f} : C \to B^A$. Then

$$\bar{\bar{f}} = ev \circ (\bar{f}, 1_A) = f$$

Similarly, if we start with $g : C \to B^A$ we have

$$\bar{g} = ev \circ (g, 1_A)$$

Then $\bar{\bar{g}}$ is defined as the unique map such that

$$\bar{g} = ev \circ (\bar{\bar{g}}, 1_A)$$

And so since $\bar{\bar{g}}$ is unique, $\bar{\bar{g}} = g$. $\square$

The fact that morphisms $f : C \times A \to B$ correspond one to one with morphisms $\bar{f} : C \to B^A$ also reveals some of what exponentials are like in a categorical poset. Let X be a poset with binary products and $x, y \in X$. Then the exponential $y^x := x \to y$ is that element such that for any $z \in X$, $z \wedge x \leq y$ if and only if $z \leq x \to y$. Thus a poset has exponentials if it is a meet-semilattice equipped with an implication operation. Similarly a set of formulas preordered by classical consequence has exponentials. For any two formulas φ and ψ, the exponential ψ^φ is simply the material conditional $\neg\varphi \vee \psi$, since $\chi \wedge \varphi \vdash \psi$ if and only if $\chi \vdash \neg\varphi \vee \psi$.

With all of these notions in place we come to the main definition.

Definition 1.6. *Let $\mathcal{C}$ be a category. Then $\mathcal{C}$ is **Cartesian closed** if and only if it has a terminal object, and any two objects have both a product and an exponential.*

We've thus already shown that **Set** is Cartesian closed. We've also seen that a bounded meet-semilattice equipped with an implication operation is Cartesian closed. Thus all Boolean algebras and Heyting algebras can be viewed as Cartesian closed categories. The goal going forward will be to argue that Cantor's theorem flows merely from the Cartesian closedness of **Set** and not from anything like the distribution of sizes of sets. To make this argument we'll first show how cantor's theorem can be viewed as an immediate corollary of a theorem that holds in any Cartesian closed category. It will prove to be very helpful to first go over some distintive feature of Cartesian closed categories.

1.1. **Points, names and diagonals.** In what follows we'll fix a cartesian closed category $\mathcal{C}$ with terminal object $1 \in \mathcal{C}$. In general, we cannot assume the objects of $\mathcal{C}$ are sets, and so cannot assume that it makes sense to speak of "elements" of objects in $\mathcal{C}$. There is however a generalized notion of element that can be defined. For any object A we say that a morphism $p : 1 \to A$ is a ***point*** in A. The guiding intuition here is the category **Set**. For any set A, there is a bijection between A and the set of points of A that maps each point $p : 1 \to A$ to the unique $x \in p(1)$ (the image of 1 under p). The result is that in **Set** we can translate between talk of elements of a given set and talk of points in a set.

Equipped with this notion of point, we can also make clearer the sense in which the exponential of two objects can be thought of as something like an abstract function space. First we introduce the following abbreviation. For any morphism $f : A \times B \to C$ and any point $a : 1 \to A$, we define the morphism $f(a, \cdot) : B \to C$ as the following composite:

$$\begin{tikzcd} 1 \times B \arrow[r, "(a, 1_B)"] & A \times B \arrow[d, "f"] \\ B \arrow[u, "\pi_B^{-1}"] \arrow[r, "f(a,\cdot)"'] & C \end{tikzcd}$$

Return to the category **Set**. Let $a_0 \in A$ be the element corresponding to a. Then for any $b \in B$ we have that $f(a_0, b) = f(a, \cdot)(b)$. So we can think of $f(a, \cdot)$ as the unary function that results from "plugging up" one of the slots in the binary function f.

With this notation in place we'll show that there is a bijection from the set of morphisms $\mathcal{C}(A, B)$ from A to B and the set of point $\mathcal{C}(1, B^A)$ in B^A. First we show the following lemma.

Lemma 1.7. *Let $\mathcal{C}$ be a cartesian closed category and let $f : A \to B$ be a morphism. Then there is a unique point $\ulcorner f \urcorner : 1 \to B^A$ such that*

$$ev(\ulcorner f \urcorner, \cdot) = f$$

Proof. What needs to be shown is that there is a unique point $\ulcorner f \urcorner : 1 \to B^A$ such that

$$\begin{tikzcd} 1 \times A \arrow[r, "(\ulcorner f \urcorner, 1_A)"] & B^A \times A \arrow[d, "ev"] \\ A \arrow[u, "\pi_A^{-1}"] \arrow[r, "f"'] & B \end{tikzcd}$$

Let $\ulcorner f \urcorner = \overline{f \circ \pi_A}$ (i.e. the transpose of the composite $f \circ \pi_A$). By definition, this map is the unique map such that

$$\begin{tikzcd} B^A \times A \arrow[r, "ev"] & B \\ 1 \times A \arrow[u, "(\overline{f \circ \pi_A}, 1_A)"] \arrow[ur, "f \circ \pi_A"'] & \end{tikzcd}$$

commutes. And so since $\pi_A : 1 \times A \to A$ is an isomorphism, $\overline{f \circ \pi_A}$ is also the unique map for which

$$\begin{tikzcd}
B^A \times A \arrow[r, "ev"] & B \\
1 \times A \arrow[u, "(\overline{f \circ \pi_A}, 1_A)"] \arrow[ur, "f \circ \pi_A"'] & \\
A \arrow[u, "\pi_A^{-1}"] &
\end{tikzcd}$$

commutes. Since π_A and π_A^{-1} cancel out, the following commutes:

$$\begin{tikzcd}
1 \times A \arrow[r, "f \circ \pi_A"] & B \\
A \arrow[u, "\pi_A^{-1}"] \arrow[ur, "f"'] &
\end{tikzcd}$$

Then stringing these together tells us that $\overline{f \circ \pi_A}$ is the unique maps such that

$$\begin{tikzcd}
1 \times A \arrow[r, "(\overline{f \circ \pi_A}, 1_A)"] & B^A \times A \arrow[d, "ev"] \\
A \arrow[u, "\pi_A^{-1}"] \arrow[r, "f"'] & B
\end{tikzcd}$$

commutes.

□

We call the point $\ulcorner f \urcorner : 1 \to B^A$ the ***name*** of the morphism $f : A \to B$. The map that sends each morphism to its name describes the relevant bijection.

Proposition 1.8. *For each A and B the function*

$$f \mapsto \ulcorner f \urcorner : \mathcal{C}(A, B) \to \mathcal{C}(1, B^A)$$

is a bijection.

For any point $p : 1 \to A$ and morphism $f : A \to B$ we let $f(p)$ abbreviate $f \circ p$. For a morphism $(f, g) : C \to A \times B$ and morphism $h : A \times B \to D$ we write $h(f, g)$ for $h \circ (f, g)$. With this notation in place we can prove the following important lemma.

Lemma 1.9. *Let $f : A \times A \to B$ be a morphism and $p : 1 \to A$ a point. Then*

$$f(p,p) = ev(\bar{f}(p), p)$$

where $\bar{f} : A \to B^A$ is the transpose of f.

Proof. What needs to be show is that the following diagram commutes.

$$\begin{tikzcd}
1 \arrow[r, "(\bar{f}(p),p)"] \arrow[d, "(p,p)"'] & B^A \times A \arrow[d, "ev"] \\
A \times A \arrow[r, "f"'] & B
\end{tikzcd}$$

We first note that the following commutes

$$\begin{tikzcd}
A & A \times A \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & A \\
& A \arrow[u, "\Delta"] \arrow[ul, "1_A"] \arrow[ur, "1_A"'] & \\
& 1 \arrow[u, "p"] \arrow[uul, bend left, "p"] \arrow[uur, bend right, "p"'] &
\end{tikzcd}$$

Hence $\Delta(p) = (p,p)$. Since the following commutes:

$$\begin{tikzcd}
A & & A \times A \arrow[ll, "\pi_{A_1}"'] \arrow[rr, "\pi_{A_2}"] & & A \\
& 1 \arrow[ul, "p"] & 1 \times 1 \arrow[u, "p \times p"] \arrow[l, "\pi_{1_1}"'] \arrow[r, "\pi_{1_2}"] & 1 \arrow[ur, "p"'] & \\
& & 1 \arrow[u, "\Delta"] \arrow[ul, "1_1"] \arrow[ur, "1_1"'] \arrow[uull, bend left, "p"] \arrow[uurr, bend right, "p"'] & &
\end{tikzcd}$$

we can then conclude that $\Delta(p) = p \times p \circ \Delta$. Hence

$$f(p,p) = f \circ p \times p \circ \Delta$$

By the definition of the transpose of f, we can then infer

$$f(p,p) = ev \circ (\bar{f} \times 1_A) \circ (p \times p) \circ \Delta$$

Thus to prove the result it suffices to show that

$$(\bar{f} \times 1_A) \circ (p \times p) \circ \Delta = (\bar{f}(p), p)$$

This fact can be verified by noting first that the following diagram commutes (suppressing now the relevant projections):

$$\begin{tikzcd}
B^A & B^A \times A \arrow[l] \arrow[r] & A \\
A \arrow[u, "\bar{f}"] & A \times A \arrow[l] \arrow[r] \arrow[u, "\bar{f} \times 1_A"'] & A \arrow[u, "1_A"'] \\
1 \arrow[u, "p"] & 1 \times 1 \arrow[l] \arrow[r] \arrow[u, "p \times p"'] & 1 \arrow[u, "p"'] \\
 & 1 \arrow[ul, "1_1"'] \arrow[u, "\Delta"] \arrow[ur, "1_1"] &
\end{tikzcd}$$

Thus, reading off the diagram,

$$\begin{aligned}
\bar{f} \times 1_A \circ p \times p \circ \Delta &= (\bar{f} \circ p \circ 1_1, 1_A \circ p \circ 1_1) \\
&= (\bar{f}(p), p)
\end{aligned}$$

□

2. Diagonalization and Cartesian Closed Categories

In this section, I will state and prove Lawvere's fixed point theorem for Cartesian closed categories, and the illustrate a couple of its applications.

Let $\mathcal{C}$ be a Cartesian closed category. We say that a morphism $f : A \to B$ is ***point-surjective*** if for every point $q : 1 \to B$ there is a point $p : 1 \to A$ such that $f(p) = q$. We say that a morphism $g : B \to B$ has a ***fixed point*** if there is a point $q : 1 \to B$ such that $g(q) = q$.

Theorem 2.1 (Lawvere's fixed point theorem)**.** *Let $\mathcal{C}$ be a cartesian closed category. Then a morphism $f : A \to B^A$ is point- surjective only if every morphism $g : B \to B$ has a fixed point.*

Proof. Suppose that $f : A \to B^A$ is point-surjective and let $g : B \to B$ be an arbitrary morphism. Defin Φ to be the following composite:

$$\begin{tikzcd} A \times A \arrow[r, "\bar{f}"] & B \arrow[d, "g"] \\ A \arrow[u, "\Delta"] \arrow[r, "\Phi"'] & B \end{tikzcd}$$

Let $\ulcorner \Phi \urcorner : 1 \to B$ name Φ. Since f is point-surjective, there is a point $p : 1 \to A$ such that $f(p) = \ulcorner \Phi \urcorner$. Thus

$$\begin{aligned} \bar{f}(p,p) &= ev(f(p), p) && \text{from Proposition 1.5 and Lemma 1.9} \\ &= ev(\ulcorner \Phi \urcorner, p) && \text{by definition} \\ &= \Phi(p) && \text{Lemma 1.7} \\ &= g(\bar{f}(p,p)) && \text{by definition} \end{aligned}$$

□

3. Applications

Coming soon!